Entry Task:

Give a Taylor series answer for

$$\int_0^1 e^{-t^2}\,dt$$

RECALL

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \quad \text{for all } z.$$

THUS,

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{1}{k!}(-t^2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k}$$

SO

$$\int e^{-t^2}\,dt = C + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(2k+1)} t^{2k+1}$$

$$\Longrightarrow \int_0^1 e^{-t^2}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} t^{2k+1}\Big|_0^1$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)}}$$

HERE ARE THE FIRST FEW TERMS:

$$\int_0^1 e^{-t^2}\,dt = \frac{1}{0!(1)} - \frac{1}{1!(3)} + \frac{1}{2!(5)} - \frac{1}{3!(7)} + \frac{1}{4!(9)} - \frac{1}{5!(11)} + \cdots$$

$0.\overline{6}$

$0.7\overline{6}$

0.74286

0.747487

0.746729

OUT TO $k=10$ GIVES

0.7468241338

MATCHES ACTUAL ANSWER

Example (*from HW*)

Write down the Taylor series for $\sin(t)$ based at 0. Then use it to give the Taylor series for

$$A(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

$$\sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots$$

$$\frac{\sin(t)}{t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k} = 1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^4 - \cdots$$

$$\Rightarrow \int_0^x \frac{\sin(t)}{t}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{(2k+1)} t^{2k+1}\Big|_0^x = x - \frac{1}{3!}\frac{1}{3}x^3 + \frac{1}{5!}\frac{1}{5}x^5 - \frac{1}{7!}\frac{1}{7}x^7 + \cdots$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!\,(2k+1)} x^{2k+1}}$$

1. Bring some form of photo ID. We will check ID's.

2. Final grades will be posted by Friday of next week.

3. Allowed:
   (a) One 8.5 by 11 inch sheet of handwritten notes (front and back)
   (b) Ti-30x IIS calculator (this model only)

4. ***Coverage***

Eight pages of questions.

Exam is comprehensive (covers everything).

Quick Review:
Ch. 12: 3D Basics (vector facts, lines, planes, basic surfaces, ...)

Ch 13: 3D Curves (accel/vel/position, tangent vector, unit tangent, tangent line, normal vector, curvature, ...)

Ch 14/15: 3D Surfaces (traces, partial deriv, max/min, double integrals, ...)

TN: Taylor Polynomials and Series (use deriv. to find Taylor Poly., error bounds, Taylor series patterns, ...)

A Recent Final Question on
Taylor Polynomials and Series

Winter 2018 / Problem 7

Let $f(x) = x^2 \sin(x^3) + \frac{1}{8-x^3}$.

(a) Find the 6th Taylor polynomial based at 0. GO TO x^6!

(b) Give the open interval of convergence of the Taylor series for f(x) based at 0.

(a) $x^2 \sin(x^3)$

$$= x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x^3)^{2k+1} = x^2\left((x^3) - \frac{1}{3!}(x^3)^3 + \frac{1}{5!}(x^3)^5 - \dots\right.$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{6k+5} = \underbrace{x^5}_{\text{STOP}} - \frac{1}{3!}x^{11} + \frac{1}{5!}x^{17} \dots \quad \boxed{\text{FOR ALL } x}$$

$$\frac{1}{8-x^3} = \frac{1}{8}\left(\frac{1}{1-\frac{1}{8}x^3}\right) \quad \boxed{\text{For } -1 < \frac{1}{8}x^3 < 1}$$

$$= \frac{1}{8}\sum_{k=0}^{\infty}\left(\frac{1}{8}x^3\right)^k$$

$$= \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{8^k}x^{3k} = \frac{1}{8}\left(\underbrace{1 + \frac{1}{8}x^3 + \frac{1}{64}x^6}_{\text{STOP!}} + \dots\right.$$

THUS

$$x^2\sin(x^3) + \frac{1}{8-x^3} \approx x^5 + \frac{1}{8} + \frac{1}{64}x^3 + \frac{1}{512}x^6$$

$$\boxed{T_6(x) = \frac{1}{8} + \frac{1}{64}x^3 + x^5 + \frac{1}{512}x^6}$$

(b) $-1 < \frac{1}{8}x^3 < 1$

$\Rightarrow -8 < x^3 < 8$

$\Rightarrow \boxed{-2 < x < 2}$

Winter 2018 / Problem 8

Let $g(x) = \sqrt{3 + x^2}$.

(a) Find the 1st Taylor polynomial based at 1.

(b) Give a bound on the error over the interval [0.5, 1.5].

(a) $g(x) = (3+x^2)^{1/2} \Rightarrow g(1) = 2$

$g'(x) = \frac{1}{2}(3+x^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{3+x^2}} \Rightarrow g'(1) = \frac{1}{2}$

$$\boxed{T_1(x) = 2 + \frac{1}{2}(x-1)}$$

(b) $g''(x) = (3+x^2)^{-1/2} \cdot (1) - \frac{1}{2}(3+x^2)^{-3/2}\, 2x \cdot x$

$= \frac{1}{\sqrt{3+x^2}} - \frac{x^2}{(3+x^2)^{3/2}}$

COMMON DENOMINATOR

$\frac{3+x^2}{(3+x^2)^{3/2}} - \frac{x^2}{(3+x^2)^{3/2}}$

$g''(x) = \frac{3}{(3+x^2)^{3/2}}$

$|g''(x)| = \frac{3}{(3+x^2)^{3/2}} \leq \frac{3}{(3+0.5^2)^{3/2}}$

[0.5, 1.5]

$M = \frac{3}{(3.25)^{3/2}} \approx 0.51203$

$\text{Error} \leq \frac{M}{2!}|x-1|^2$

$\leq \frac{0.51203}{2}|0.5-1|^2$

$= \boxed{0.064}$